NPTEL web course on Complex Analysis

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Module: 8: **Mapping of Elementary transformation**
Lecture: 3: **The mapping $w = z^2$ and its inverse mapping**

The mapping $w = z^2$

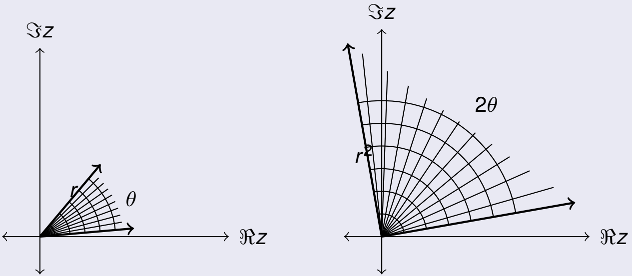
The mapping $w = z^2$

In this section we discuss the mapping $w = z^2$.

The mapping $w = z^2$

- The image of a point in the z-plane under the map $w = z^2$ is best visualized by writing the complex number in its polar form.
- For, if $z = re^{i\theta}$, $w = r^2 e^{i2\theta}$. This implies a magnification followed by a rotation of z by its argument in the w-plane.
- Hence an angle of θ radians will be mapped to an angle of 2θ radians with the length of the rays stretching or shrinking according as $r < 1$ or $r > 1$.
- The transformation for $r < 1$ is shown below.

The mapping $w = z^2$



The mapping $w = z^2$

Now we discuss some particular cases for the mapping $w = z^2$.

The mapping $w = z^2$

Transformations of the co-ordinate axes

- To study the mapping properties, we write $z = x + iy$ so that

$$u = x^2 - y^2, \quad v = 2xy$$

- Consider the real axis $y = 0$, $-\infty < x < \infty$.
- Then $u = x^2$ and $v = 0$, which implies $u \geq 0$. As x varies from $-\infty$ to 0, u varies from ∞ to 0 and then as x moves further away from 0, u traverses back the real axis from 0 to ∞.
- Similarly for the imaginary axis $x = 0$, $-\infty < y < \infty$, $u = -y^2$ and $v = 0$.
- So as y traverses the imaginary axis from below towards origin, u traverses the negative real axis from $-\infty$ to 0 and then as y moves upwards away from origin, u traverses back the negative real axis from origin towards $-\infty$.

The mapping $w = z^2$

Image of a vertical line

- Consider the vertical line $x = k_1$ so that $u = x^2 - y^2, \quad v = 2xy$ gives $u = k_1^2 - y^2, \quad v = 2k_1 y$.
- Eliminating y, we get the parabola

$$v^2 = 4k_1^2(k_1^2 - u)$$

with vertex at $(k_1^2, 0)$ and focus at (0,0).

- Hence all vertical lines in the z-plane will have vertex on the positive real axis and focus at (0,0).

The mapping $w = z^{1/2}$

The mapping $w = z^{1/2}$

- In this section, we discuss the properties of the mapping $w = z^{1/2}$.
- Even though, this is inverse of the mapping $z = w^2$, there is an underlying difference between these two mappings.

The mapping $w = z^{1/2}$

- $w = z^{1/2}$ gives two square roots of z, when $z \neq 0$.
- In polar form, we have

$$z^{1/2} = \sqrt{r} \exp \frac{i\theta + wk\pi}{2}, \qquad k = 0, 1.$$

- Here the principal root occurs when $k = 0$.
- $z^{1/2}$ can also be written as

$$z^{1/2} = \exp\left(\frac{1}{2} \log z\right).$$

The mapping $w = z^{1/2}$

- The principal branch of the double valued function $z^{1/2}$ is obtained by taking the principal branch of $\log z$.
- We denote this principal branch of $z^{1/2}$ as F_0.
- This gives

$$F_0(z) = \exp\left(\frac{1}{2} Logz\right), \qquad |z| > 0, -\pi < Argz < \pi.$$

- The ray $\theta = \pi$ is the branch cut of F_0 and $z = 0$ is the branch point.
- Even though the values along the branch cut can be defined for F_0, they are not even continuous there.

The mapping $w = z^{1/2}$

- The particular cases of this mapping can be obtained by using the mapping $w^2 = z$.
- In that case, only the Principal function F_0 is taken into consideration.

Example

For example, the function $w = z^2$ maps the hyperbola $2xy = 1$ of the z-plane onto the line $v = 1$ in the w-plane. Hence the mapping $w = z^{1/2}$ (the principal branch, with branch cut at $\theta = 0$) maps the line $y = 1$ in the z-plane onto the branch of the hyperbola $2uv = 1$ which lies in the first quadrant of the w-plane. Both the mappings are one to one.