NPTEL web course on Complex Analysis

A. Swaminathan
I.I.T. Roorkee, India

and

V.K. Katiyar
I.I.T. Roorkee, India

Module: 8: **Mapping of Elementary transformation**
Lecture: 2: **The mapping** $w = 1/z$

The mapping $w = 1/z$

The mapping $w = 1/z$

In this section we discuss the mapping $w = 1/z$.

The mapping $w = 1/z$

- If a point $z = x + iy$ is mapped to a point $w = u + iv$ under the map $1/z$, we have
$$w = \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$
so that
$$u = \frac{x}{x^2 + y^2} \quad \text{and} \quad v = \frac{-y}{x^2 + y^2}$$
- Further under inversion, $z = 1/w$ so that
$$x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = \frac{-v}{u^2 + v^2}$$

We first discuss the transformation of the circles under the mapping $w = 1/z$.

Transformation of circles

- The general equation of circles and lines in the complex plane is given by

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

  where $A = 0$ for a line and $A \neq 0$ for a circle.

- The above equation can be re-written as

$$x^2 + y^2 + \frac{B}{A}x + \frac{C}{A}y + \frac{D}{A} = 0, \quad A \neq 0$$

- This after simplification gives

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2 + C^2 - 4AD}}{2A}\right)^2$$

The mapping $w = 1/z$

Transformation of circles

- So for $A \neq 0$, the equation represents a circle with center $\left(\frac{-B}{2A}, \frac{-C}{2A}\right)$ and radius $\left(\frac{\sqrt{B^2+C^2-4AD}}{2A}\right)^2$, provided $B^2 + C^2 > 4AD$.
- It is easy to find the image of circles in the w-plane, which is given by
$$D(x^2 + y^2) + Bu - Cv + A = 0$$
which represents in the w-plane a line if $D = 0$ and a circle if $D \neq 0$.
- This gives rise the following four cases.

The mapping $w = 1/z$

Transformation of circles

$A = 0, D = 0$ A line through origin in the z-plane is mapped to a line through origin in the w-plane.

$A \neq 0, D = 0$ A circle through origin in the z-plane is mapped to a line not passing through origin in the z-plane.

$A = 0, D \neq 0$ A line not passing through origin in the z-plane is mapped to a circle through origin in the w-plane.

$A \neq 0, D \neq 0$ A circle not passing through origin in the z-plane is mapped to a circle not passing through origin in the w-plane.

The mapping $w = 1/z$

We next discuss the transformation of the Images of lines parallel to axes under the mapping $w = 1/z$.

Image of parallel lines

- Consider the vertical line $x = k_1, k_1 \neq 0$.
- It is transformed into a circle through origin in the w-plane.
- For this line, the given equation of circles gives $A = 0, B = 1, C = 0$ and $D = -k_1$.
- This implies the image is $-k_1(u^2 + v^2) + u = 0$ or the circle

$$\left(u - \frac{1}{2k_1}\right)^2 + v^2 = \frac{1}{(2k_1)^2}$$

  with center at $(1/2k_1, 0)$ and radius $1/2k_1$.
- The transformation of a family of lines parallel to the imaginary axis ($k_1 > 0$) is shown below.

The mapping $w = 1/z$

Image of parallel lines

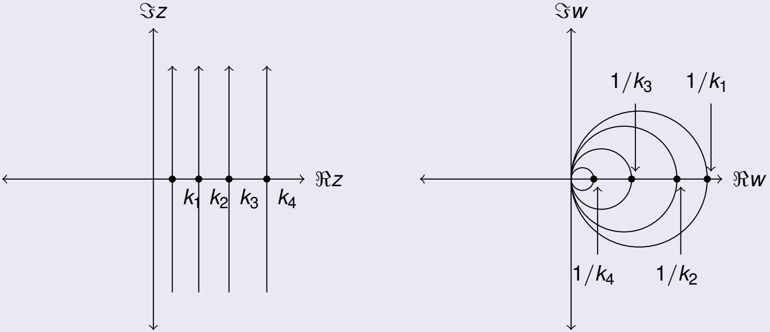


Image of parallel lines

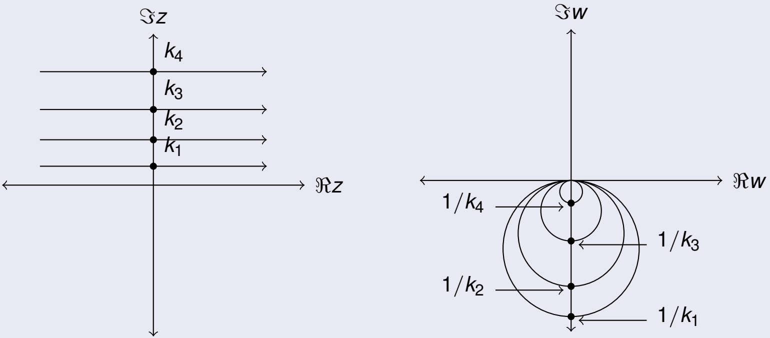
- Similarly, a horizontal line $y = k_2, k_2 \neq 0$ is mapped to the circle

$$u^2 + \left(v + \frac{1}{2k_2}\right)^2 = \frac{1}{(2k_2)^2}$$

  with center at $(0, -1/2k_2)$ and radius $1/2k_2$.
- The transformation of a family of lines parallel to the real axis $(k_2 > 0)$ is shown below.

The mapping $w = 1/z$

Image of parallel lines



The mapping $w = 1/z$

Example

Find the image of the disc $|z-1|^2 \leq 1$.

Solution.

- The given circle $|z-1|^2 = 1$ passes through origin and hence is transformed onto a line not passing through origin.
- The given circle is
$$x^2 + y^2 - 2x = 0$$
so that we have $A = 1, B = -2, C = 0, D = 0$ and then the equation of image circle implies the image $u = 1/2$ in the w-plane.
- Further under inversion any point $0 < x < 2$ will be mapped to $u > 1/2$.
- Hence the image of the given disc will be the half plane $u \geq 1/2$.

The mapping $w = 1/z$