NPTEL web course on Complex Analysis

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Module: 8: **Mapping of Elementary transformation**
Lecture: 4: **The mapping** $w = \sin z$

The mapping $w = \sin(z)$

The mapping $w = \sin z$

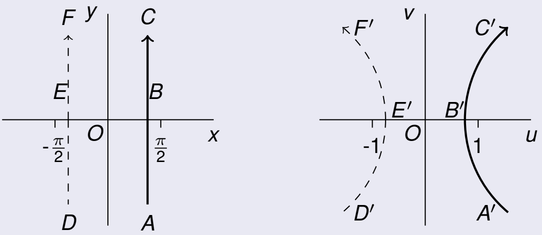
In this section, we study the properties of the mapping $w = \sin z$.

The mapping $w = \sin z$

- We use the relation $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$ to get
$$u = \sin x \cosh y, \quad v = \cos x \sinh y.$$
- Consider now the vertical line $x = a$.
- We assume for the moment that $a \neq -\pi/2, 0, \pi/2$. Then
$$u = \sin a \cosh y, \quad v = \cos a \sinh y$$
- Eliminating y, we arrive at the image curve
$$\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1$$
which is a hyperbola.
- The above equation is valid because of the restriction on a, so that $\sin a \neq 0$ and $\cos a \neq 0$.

- This hyperbola intersects the real axis at $(\pm \sin a, 0)$ and has foci at $(\pm 1, 0)$. Suppose now $0 < a < \pi/2$.
- Since the image of the point $(a, 0)$ is $(\sin a, 0)$, the branch of hyperbola passing through $(-\sin a, 0)$ cannot be part of the image curve.
- Hence the line $x = a$ is mapped to the right branch of the hyperbola.
- For example, the vertical lines $x = \pi/3$ and $x = \pi/6$ are mapped to the branches of hyperbola (with focus at (1,0)) passing through $(\sqrt{3}/2, 0)$ and $(1/2, 0)$ respectively.
- But the image of the point $(-a, 0)$ is $(-\sin a, 0)$ so that the line $x = -a$ is mapped to the left branch of the hyperbola $\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1$.

The mapping $w = \sin z$



The mapping $w = \sin z$

Special cases

- We now discuss the cases for the special values of a.
- If $a = 0$, that is the imaginary axis, we have $u = 0$ and $v = \sinh y$. As y varies from $-\infty$ to ∞, v too varies with it.
- Hence the imaginary axis in the z-plane is mapped to the imaginary axis in the w-plane.

The mapping $w = \sin z$

Special cases

- The line $x = \pi/2$ is mapped to curve $u = \cosh y$, $v = 0$.
- Now as y varies from $-\infty$ to 0, $\cosh y$ being an even function, u travels from ∞ to 1 and then as y moves upwards from 0, u moves away from 1 towards ∞.
- Hence the image curve is $u \geq 1$. Similarly, the line $x = -\pi/2$ is mapped to $u \leq -1$.

Special cases

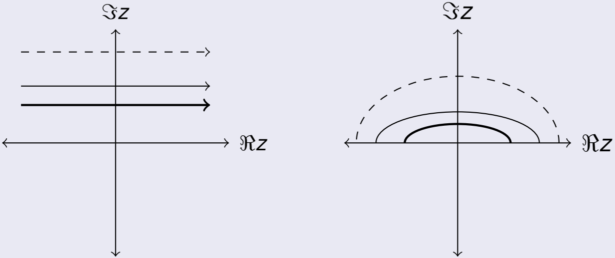
- For the transformation of the horizontal line $y = b$, $-\pi/2 \leq \pi/2$, we proceed exactly as above.
- In this case, $u = \sin x \cosh b$ and $v = \cos x \sinh b$ from which we obtain the image curve as

$$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1$$

  which is an ellipse with intercepts at $(\pm \cosh b, 0)$ and $(0, \pm \sinh b)$.
- If $b > 0$, the image of the line $y = -b$ will be the upper half of the ellipse while that of $y = -b$ will be the lower half.
- In particular the real axis ($b = 0$) will be mapped to the curve $u = \sin x$, $v = 0$ which is simply the interval $-1 \leq u \leq 1$.

The mapping $w = \sin z$

Special cases

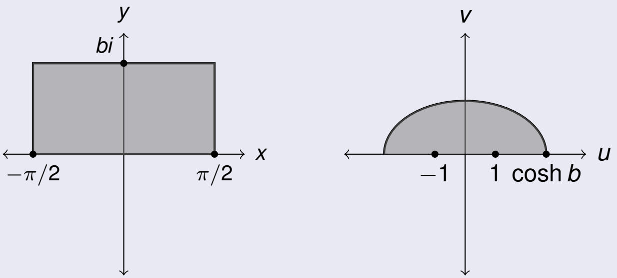


The mapping $w = \sin z$

The above transformations of vertical and horizontal lines show that the rectangular region $-\pi/2 \leq x \leq \pi/2$, $0 \leq y \leq b$ is mapped under $w = \sin z$ to the upper half of an elliptical region with center at origin.

The mapping $w = \sin z$

Transformation of a rectangular region



The mapping $w = \sin z$

Special cases

- The vertical lines $x = -\pi/2$ and $x = \pi/2$ are respectively mapped to $u \leq -1$ and $u \geq 1$, while the line segment $-\pi/2 \leq x \leq \pi/2$ is mapped to the interval [-1,1].
- The line $y = b$ forms the upper boundary of the elliptic region intersecting the u-axis at $(\pm \cosh b, 0)$.

The mapping $w = \sin z$

Example

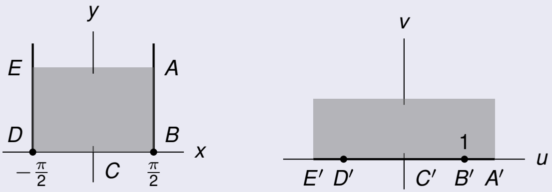
Find the image of the semi-infinite strip

$$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad 0 \leq y < \infty$$

Solution. The transformation is shown below.

The mapping $w = \sin z$

Image of the semi-infinite strip



Example

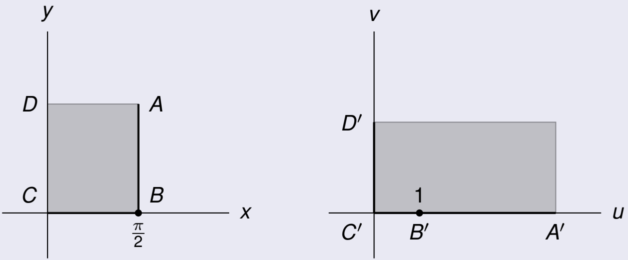
Find the image of the semi-infinite strip

$$0 \leq x \leq \frac{\pi}{2}, \quad 0 \leq y < \infty$$

Solution. The transformation is shown below.

The mapping $w = \sin z$

Image of the semi-infinite strip



Since $\cos z = \sin(z + \pi/2)$, $\sin hz = i \sin(iz)$ and $\cos hz = \cos(iz)$, these mappings can be studied as a special case of the transformation $w = \sin z$ using elementary transformation such as translation, rotation etc.