NPTEL web course on Complex Analysis

A. Swaminathan

I.I.T. Roorkee, India

and

V.K. Katiyar

I.I.T. Roorkee, India

Module: 2: **Functions of a Complex Variable**
Lecture: 3: **Limit, Continuity and Differentiability**

Functions of a complex variable

Functions of a Complex variable

Definition

A *function* f is a rule that assigns each element x in a set X one and only one element y in a set Y. We write this as

$$y = f(x)$$

and call y the image of x under f.

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Notation

Symbolically, we denote this as

$$f : X \to Y.$$

Here X is called the domain of definition (which is different from the definition of domain) and B called the range. Note that $f(X) \subseteq Y$.

Definition

We are interested in considering the domain of definition and range are subsets of the complex plane. Then f is called complex valued function of a complex variable.

Functions of a Complex variable

Definition

Let $f : D_1 \subseteq \hat{\mathbb{C}} \longrightarrow D_2 \subseteq \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$, which includes point at infinity is $\mathbb{C} \cup \{\infty\}$. Then for any point $z \in D_1$, f maps (assigns) a point $w \in D_2$ such that $w = f(z)$.

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- If $z = x + iy$ and $w = u + iv$, then $u \equiv u(x, y)$ and $v \equiv v(x, y)$.
- Further, D_2 may be in $\mathbb{R}$, but this case may not be interesting.
- If f is real valued function, then f is well-defined $\iff$ the domain and range are well-defined.

Example

- $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \sqrt{x}$ is not a function, well-defined, so it has to be modified as $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $f(x) = |\sqrt{x}|$, to make it well-defined.

  Similarly, in the complex case also a function f has to be well-defined.
- Let $f(z) = 1/(z^n - 1)$. Then f is defined for all values except the n-th roots of unity.
- $f : \mathbb{C} \longrightarrow \mathbb{C}$, by $f(z) = z^{1/n}$, $n \in \mathbb{N}$.
  To get f well-defined, it may be said as "$f(z) =$ one of the values of $z^{1/n}$".

Note.

The advantage of $f : \mathbb{R} \longrightarrow \mathbb{R}$ is, it can be visualized in a plane. This is not true for f defined over $\mathbb{C}$. In the case of $\mathbb{C}$, the range can not be explicitly described for any f. Only when f is defined over a particular domain (of definition) the image for that domain can be exhibited.

Example

Example of a real valued function Let $f(x) = \sin x$. Then we know that $f(x)$ is defined for all x over the real line and the image is a periodic function with period 2π.

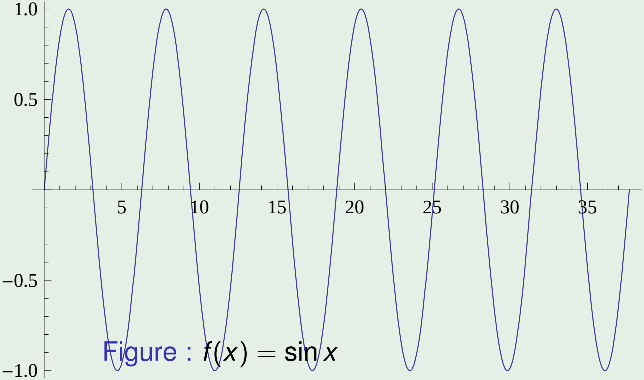

Figure : $f(x) = \sin x$

Example

Corresponding complex valued function Let $f(z) = \sin z$. Nothing can be said about the range. But if a particular domain of definition is given, like $x =$constant, then the range can be obtained. In this case, if

$$f(z) = u + iv = \sin z = \sin x \cosh y + i \cos x \sinh y.$$

This implies

$$u = \sin x \cosh y, \qquad v = \cos x \sinh y.$$

Since $x = c$, this implies $\dfrac{u}{\alpha_1} = \cosh y$ and $\dfrac{v}{\alpha_2} = \sinh y$, where $\alpha_1 = \sin c$, $\alpha_2 = \cos c$. This gives

$$\frac{u^2}{{\alpha_1}^2} - \frac{v^2}{{\alpha_2}^2} = 1 \qquad \text{(an hyperbola)}.$$

Example

Corresponding complex valued function

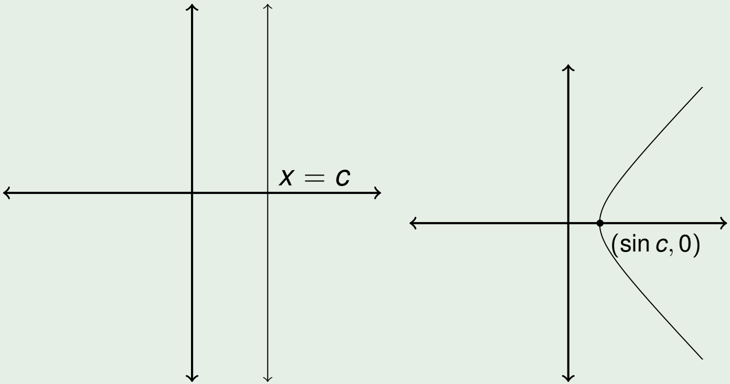

Figure : $f(z) = \sin z$

Limit and continuity

Definition of a complex sequence

The sequence of complex numbers is similar to the sequence of real numbers, except that it takes complex values. Let $\{z_n\}_{n=1}^{\infty}$ be a complex sequence.

Convergence of a complex sequence

Definition

The sequence $\{z_n\}_{n=1}^{\infty}$ converges to a limit l if given $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ (sufficiently large), so that

$$|z_n - l| < \epsilon \qquad \text{whenever} \quad n > N_0.$$

Convergence of a complex sequence

Example

Let $\{z_n\}_{n=1}^{\infty}$ be the sequence $z_n = \left(\frac{i}{3}\right)^n$. Then

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \left(\frac{i}{3}\right)^n \to 0, \qquad \text{for} \qquad n > \frac{\log\frac{1}{3}}{\log(\epsilon)}.$$

Here we write $\lim_{n\to\infty} (i/3)^n = 0$.

Example

$z_n = \dfrac{2+3n}{1+in}$. Then $\lim\limits_{n\to\infty} z_n = -3i$.

Convergence of a complex sequence

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Example

$z_n = i^n$. Then $\lim_{n\to\infty} (i)^n \nrightarrow$ unique value, which implies limit does not exist.

Limit of a function

Assumptions

- Whenever we define $f : D_1 \to D_2$, D_1 and D_2 are considered as subsets of $\mathbb{C}$.
- f is assumed as well-defined.
- $z_0 = x_0 + iy_0 = (x_0, y_0)$.
- $f(z) = u + iv = u(x, y) + iv(x, y)$.

Limit of a function

Definition

Let $f : D_1 \to D_2$. let z_0 be a point in D_1. Then w_0 is limit of $f(x, y)$ at (x_0, y_0), if given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - w_0| < 0 \qquad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

Example

Question. Prove that $\lim_{z\to 1} f(z) = i/3$ where $f(z) = iz/3$ in $|z| < 1$.

Answer. Given that $|z| < 1$. Now

$$\left|\frac{iz}{3} - \frac{i}{3}\right| = \left|\frac{z-1}{3}\right|.$$

Hence, for any such z and any $\epsilon > 0$, with $\delta \leq 3\epsilon$,

$$\left|f(z) - \frac{i}{3}\right| < \epsilon \qquad \text{whenever} \quad 0 < |z-1| < \delta \leq 3\epsilon.$$

Limit of a function

Theorem

If $\lim_{z\to z_0} f(z) = L_1$ *and* $\lim_{z\to z_0} g(z) = L_2$*, then*

1. $\lim_{z\to z_0} f(z) \pm g(z) = L_1 \pm L_2,$
2. $\lim_{z\to z_0} f(z)g(z) = L_1 L_2,$
3. $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L_1}{L_2}, \quad \text{if } L_2 \neq 0.$

Continuity

Let $\lim_{z \to z_0} f(z) = L$ exists. If $L = f(z_0)$, then f is said to be continuous at $z = z_0$. This means, if $\lim_{z \to z_0} f(z) = f(z_0) \quad or \quad w_0 = f(z_0)$, then f is continuous. In the language of $\epsilon - \delta$ we have the following definition.

Continuity of a function

Definition

Let $f : D_1 \to D_2$ and $z_0 \in D$. Then $\lim_{z \to z_0} f(z) = f(z_0)$ (or f is said to be continuous) if, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \qquad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

Continuity of a function

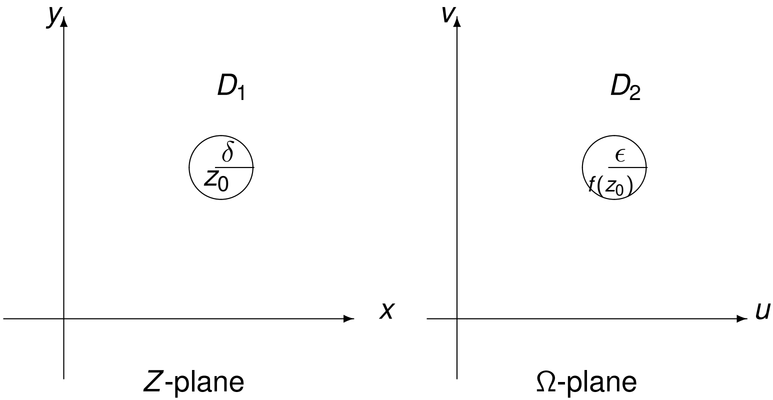

Figure : continuous mapping.

Continuity of a function

Example

To discuss the continuity of $f(z) = \dfrac{z^2+4}{z-2i}$ at $z = 2i$. Then clearly

$$\lim_{z\to 2i} \frac{z^2+4}{z-2i} = \lim_{z\to 2i} z + 2i = 4i.$$

In the $\epsilon - \delta$ language, given that $|z-2i| < \delta$, we have

$$\left| \frac{z^2+4}{z-2i} - 4i \right| = \left| \frac{(z+2i)(z-2i)}{z-2i} - 4i \right| = |z-2i| < \delta.$$

So for all $\epsilon \geq \delta$, the result is true.

Continuity of a function

Theorem

Let $f(z)$ and $g(z)$ be continuous at a point $z = z_0$. Then $f(z) \pm g(z)$ are also continuous at $z = z_0$. Also the product $f(z)g(z)$ is also continuous at $z = z_0$. But $f(z)/g(z)$ is continuous at $z = z_0$ only when $g(z_0) \neq 0$.

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Note.

The limit should approach in all possible directions in a plane including the spiral direction. Hence the understanding of these concepts for a complex variable is more challenging.

Continuity of a function

Example

- All polynomial functions $P_n(z)$ of the form

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \qquad a_i \in \mathbb{C}, \quad i = 1, 2, \ldots, n,$$

are continuous in the whole complex plane $\mathbb{C}$.

- Rational functions of the form

$$\frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n}, \qquad a_i, b_i \in \mathbb{C}, \quad i = 1, 2, \ldots, n,$$

are also continuous in the whose complex plane $\mathbb{C}$, excepts the points at which the denominator vanishes.

A known result

Theorem

A continuous image of a connected set is connected. In other words, let $f : D \to \mathbb{C}$ and D is connected. If f is continuous then $f(D)$ is connected.

Motivation

When we find the continuity of $f(z)$ in a domain D, if δ depends only on ϵ and does not depend on the point about which the continuity is discussed, then $f(z)$ is said to be *Uniformly continuous* in the domain D.

Example

Consider the function $f(z) = x^2 + iy^2$, $z = x + iy$.

- Clearly f is continuous in $\mathbb{C}$. But f is not uniformly continuous in $\mathbb{C}$.
- Consider $D_R = \{z \in \mathbb{C} : |z| < R,\ R > 0\}$.

Then f is uniformly continuous in D_R. For, consider $z_0 = x_0 + iy_0$ so that $|x - x_0| + |y - y_0| < \delta$, for some $\delta > 0$.

$$\begin{aligned} |f(z) - f(z_0)| &= |x^2 - x_0^2 + i(y^2 - y_0^2)| \\ &= |(x - x_0)(x + x_0) + i(y - y_0)(y + y_0)| \\ &\leq |(x - x_0)(x + x_0)| + |(y - y_0)(y + y_0)|. \end{aligned}$$

In D_R, $|x + x_0| \leq 2R$ and $|y + y_0| \leq 2R$. This gives

$$|f(z) - f(z_0)| \leq 2R(|x - x_0| + |y - y_0|) < 2R\delta.$$

Hence choosing $\epsilon = 2R\delta$ gives that f is uniformly continuous on D_R.

Uniform continuity

Theorem

Let $f : D \to \mathbb{C}$, f is continuous and D is compact. Then f is uniformly continuous.

Uniform continuity

Example

The function $f(z) = 1/z$ is continuous in any domain where $z \neq 0$. Consider the punctured disc $D_1 := \{z \in \mathbb{C} : 0 < |z| < R_1, R_1 > 0\}$. Then f is not uniformly continuous in D_1. But if we choose the domain as $D_2 = \{z \in \mathbb{C} : R_1 \leq |z| \leq R_2, R_1 > 0\}$, then f is uniformly continuous in D_2.