NPTEL web course on Complex Analysis

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Module: 2: **Functions of a Complex Variable**
Lecture: 7: **Harmonic functions**

Harmonic functions

Analytic functions

Higher order

If f is analytic in $D \subseteq \mathbb{C}$. then f is differentiable. Further f has derivatives of all orders (i.e. higher order derivatives such that $f', f'', f''', \ldots$ exist). This will be proved at a later stage.

Higher order

- In particular, let $f = u + iv$ be analytic in D. This implies f', f'', f''' exist.
- If f' exists, then the existence of all the first order partial derivatives u_x, u_y, v_x and u_y is trivial.
- Existence of f'' guarantees the continuity of f' and hence the continuity of u_x, u_y, v_x and u_y.
- Similarly existence of f''' guarantees the existence and continuity of f'' and in particular the existence and continuity of the second order mixed partial derivatives u_{xy} and u_{yx}.
- Because of continuity, we have $u_{xy} = u_{yx}$.
- Similar arguments lead to $v_{xy} = v_{yx}$.

Harmonic functions

- Now, f analytic implies, it satisfies the C-R equations.
- This means
$$u_x = v_y \qquad \text{and} \qquad v_x = -u_y.$$
- Now differentiating $u_x = v_y$ partially with respect to x gives $u_{xx} = v_{yx}$.
- Similarly differentiating $v_x = -u_y$ partially with respect to y gives $v_{xy} = u_{yy}$.
- Since $v_{xy} = v_{yx}$, we finally have
$$u_{xx} = -u_{yy} \implies u_{xx} + u_{yy} = 0.$$

Harmonic functions

- This second order partial differential equation is called harmonic equation and its solution is called Harmonic function.
- A parallel argument can lead to $v_{xx} + v_{yy} = 0$.
- Hence, if $f = u + iv$ is analytic in D, then u and v are harmonic in D.

Harmonic conjugates

Definition

Let $f = u + iv$ be analytic in D. Then v is called harmonic conjugate of u.

Harmonic conjugates

Problem

Given u, to find v such that $f = u + iv$ is analytic.

Harmonic conjugates

Direct method

- Given $u = u(x, y)$, find the first order partial derivatives u_x and u_y.
- Find the second order partial derivatives u_{xx}, u_{yy}. Check $u_{xx} + u_{yy} = 0$, so that u is harmonic.
- Since the corresponding f is analytic, the first order partial derivatives u_x and u_y satisfies C-R equations

$$u_x = v_y \qquad \text{and} \qquad v_x = -u_y.$$

- Hence we have v_y and $-v_x$.

Harmonic conjugates

Direct method

- Since $v = v(x, y)$, we have

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$
$$= -v_x dx + v_y dy.$$

- Substituting the values of v_x and v_y from the corresponding values of u_x and u_y gives a differential equation involving v, x and y.
- Solving this differential equation gives $v(x, y) + c$. This gives $f = u + iv + ic$ which is analytic.

Harmonic conjugates

Direct method

Example

Let $u = x^3 - 3xy^2$. Then

$$u_x = 3x^2 - 3y^2 = v_y \qquad \text{(by C-R equation) and}$$
$$u_y = -6xy = -v_x \qquad \text{(by C-R equation).}$$

Further

$$u_{xx} = 6x \qquad \text{and} \qquad u_{yy} = -6x.$$

Since $u_{xx} + u_{yy} = 6x + (-6x) = 0$, u is harmonic.

Harmonic conjugates

Direct method

Example

Now, $v = v(x, y)$ implies

$$\begin{aligned} dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \\ &= 6xydx + (3x^2 - 3y^2)dy \\ &= (6xydx + 3x^2dy) - 3y^2dx \\ &= d(3x^2y - y^3). \end{aligned}$$

Hence $v = 3x^2y - y^3 + c$. Thus
$f = u + iv = x^3 - 3xy^2 + i(3x^2y - y^3) + ic$.

Direct method

Remark

Special care has to be taken about the constants.

Harmonic conjugates

Direct method

In the above example, it is difficult to find f explicitly as a function of z, i.e., in the form $f(z) = z^3 + ic$. Even though, this method cannot give an explicit representation of f in terms of z, this method is stronger than following procedure.

Harmonic conjugates

Milne's Method

Let $f = u + iv$ be analytic. Then $f'(z)$ exists.

$$\begin{aligned} f'(z) &= u(x+iy) + iv(x+iy) \\ &= u\left(\frac{z+\bar{z}}{2} + \frac{z-\bar{z}}{2}\right) + iv\left(\frac{z+\bar{z}}{2} + \frac{z-\bar{z}}{2}\right). \end{aligned}$$

This is true for every z. Let $z = \bar{z}$

$$\begin{aligned} f(z) &= u(z,0) + iv(z,0) f'(z) \qquad = u_x + iv_x \\ &= u_x - iu_y \quad \text{(using C-R equations)} \\ &= \phi_1(x,y) - i\phi_2(x,y). \end{aligned}$$

Harmonic conjugates

Milne's Method

- Since this is true for all z, it is true for $z = \overline{z}$ also.
- This means, taking $x = z$ and $y = 0$ we have $f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$.
- Integrating this with respect to z (as in a real integral), gives

$$f(z) = \int_0^z f^{'}(z)dz = \int \phi_1(z, 0)dz - i \int \phi_2(z, 0)dz.$$

Milne's Method

Similarly

$$\begin{aligned} f^{'}(z) &= u_y + iv_y \\ &= -v_x + iv_y \qquad \text{using C-R equations} \\ &= -\psi_1(x,y) + i\psi_2(x,y) \\ &= -\psi_1(z,0) + i\psi_2(z,0). \end{aligned}$$

This gives

$$f(z) = \int f^{'}(z)dz = -\int \psi_1(z,0)dz + i\int \psi_2(z,0)dz.$$

Milne's Method

Example

Question. Find the analytic function $f(z)$ where $u(x, y) = e^x coshy$.
Answer. Using Milne's method,

$$f(z) = \int u_x(z, 0)dz - i \int u_y(z, 0)dz = \int e^z dz - i0 = e^z + c$$

implies $f(z) = e^z + c$.

A caution

Remark

- *This method will give $f(z)$ explicitly in terms of z, if either u or v is known.*
- *But the direct method gives $f(z)$ in terms of x and y and not in explicit form (in terms of z).*
- *Note that this method is for computational purpose only and the mathematical validation of this method is not guaranteed here.*