NPTEL web course on Complex Analysis

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Module: 8: **Mapping of Elementary transformation**
Lecture: 1: **The mapping** $w = \exp(z)$

Mapping of elementary transformation

In this chapter, we consider some particular transformations. For these transformations, based on the given domain, the range has specific properties. These facts will be established in the following.

Some Special Transformation

The mapping $w = \exp(z)$

The mapping $w = \exp(z)$

- It is known that $\exp z \neq 0$ for all $z \in \mathbb{C}$.
- Hence the origin in the w-plane does not belong to the set of images of the points in the z-plane under the map $\exp x$.

The mapping $w = \exp(z)$

First, we find the pre-images of the points in the w-plane for the mapping $w = \exp(z)$.

The mapping $w = \exp(z)$

Pre-images of points in the w-plane

- We show here that any $w = u + iv \neq 0$ in the w-plane is the image of some $z = x + iy$ in the z-plane.
- Fix w. Then $w = e^z \Rightarrow w = e^x e^{iy}$ so that

$$|w| = e^x \quad \text{and} \quad \arg w = y + 2k\pi, \quad k \in \mathbb{Z}$$

- In particular if we choose the principal argument of w, we have the pre-image

$$z = \ln|w| + iArgw$$

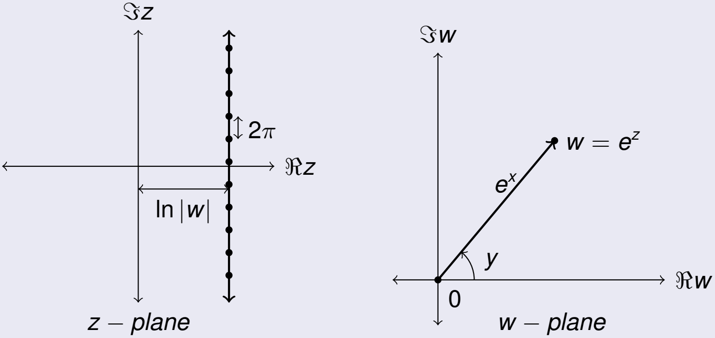
- The above relation assures the existence of at least one pre-image of any given w.
- If we now allow k to vary over $\mathbb{Z}$, we obtain the infinite set of pre-images of w:

$$z = \ln|w| + i(argw + 2k\pi)$$

The mapping $w = \exp(z)$

Pre-images of points in the w-plane

All the above points lie on a vertical line as shown below.



The mapping $w = \exp(z)$

Now we find the image of a horizontal line under the mapping $w = \exp(z)$.

The mapping $w = \exp(z)$

Image of a horizontal line

- Suppose z traces out a straight line parallel to the real axis, that is $z = t + ib$, b fixed and $t \in (-\infty, \infty)$.
- Then $w = e^t.e^{ib}$ which (in the polar form) represents a point at distance e^t from the origin with argument b.
- Since e^t is non-negative and strictly increasing, as t varies from $-\infty$ to ∞, e^t goes on increasing with the same argument (as b is fixed). This represents a ray in the w-plane from the origin, with the origin deleted, and passing through the point $(\cos b, \sin b)$.

Now we find the image of a vertical line under the mapping $w = \exp(z)$.

Image of a vertical line

- Consider the vertical line $z = a + it$, a fixed and $t \in (-\infty, \infty)$.
- Then
$$w = e^z = e^a(\cos t + i \sin t)$$
which is a circle with center as origin and radius e^a.
- Further, as t varies from 0 to 2π, w moves on the circle once in the positive direction.
- Hence as t varies from $-\infty$ to ∞, w traces out the circle infinitely many times.
- In particular, $\exp z$ wraps the imaginary axis ($a = 0$) infinitely many times around the unit circle.

The mapping $w = \exp(z)$

Now we find the image of a horizontal strip under the mapping $w = \exp(z)$.

Image of a horizontal strip

- The above discussion implies that the image of a family of lines parallel to the real axis in the z-plane is transformed into a family of rays emanating from the boundary of a neighborhood of the origin in the w-plane.
- Similarly, a family of lines parallel to the imaginary axis is mapped to a family of concentric circles with origin as the center, with the distance between the concentric circles increasing exponentially.
- The image of a horizontal strip under the exponential map is shown below.

Image of a horizontal strip

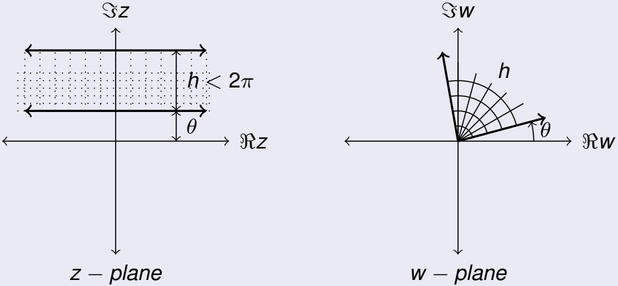


Image of a horizontal strip

- The width of the horizontal strip is taken to be less that 2π to force $\exp z$ to be one-one.
- For consider a point z_0 in the horizontal strip. For this z_0, we get a unique point in the w-plane.
- But when we pull back the same point to the Z-plane, all its pre-images will lie outside the horizontal strip if its width is less that 2π. Hence $\exp z$ becomes univalent for this particular domain.
- Further the above figure implies that if we consider a horizontal strip of width 2π and partition it into strips of width $h_1, h_2, \ldots, h_n$ so that $h_1 + h_2 + \ldots + h_n = 2\pi$, the w-plane under the exponential map will be partitioned into angles of $h_1, h_2, \ldots, h_n$ radians.
- The image of a horizontal strip of width 2π is shown below.

The mapping $w = \exp(z)$

Image of a horizontal strip

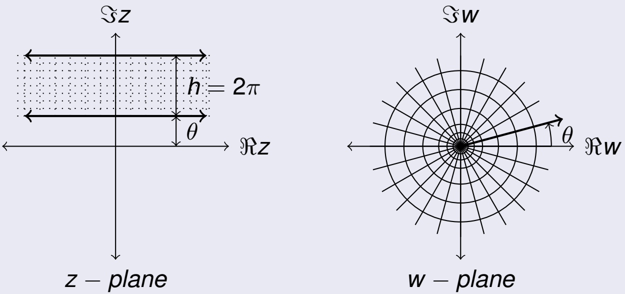


Image of a horizontal strip

Example

Find the image of the semi-infinite horizontal strip

$$\{z = x + iy \in \mathbb{C} : x \leq 0, \quad 0 \leq y \leq \pi\}$$

Solution. The image is drawn noting the following points.

- The image of the horizontal line *DE*, $x \leq 0, y = \pi$ is mapped to a ray with argument π and length 1, that is the interval [-1,0).
- The image of the vertical line *BCD*, $x = 0, 0 \leq y \leq \pi$ is mapped to the unit circle. But as y from 0 to π, w traverses the upper half of the unit circle only once.
- The segment of the negative real axis *AB* will be mapped to a ray with argument 0 and length 1, that is the interval (0,1].

The transformation is shown below.

Image of a horizontal strip

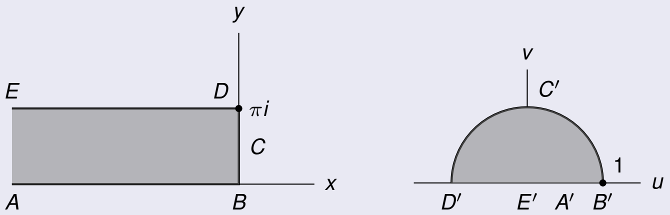


Image of a horizontal strip

Example

Find the image of the following domain under the exponential map:

$$\{z = x + iy \in \mathbb{C} : 0 < \alpha \leq x \leq \beta, \quad 0 \leq y \leq \pi i\}$$