

NPTEL web course on Complex Analysis

A. Swaminathan

I.I.T. Roorkee, India

and

V.K. Katiyar

I.I.T. Roorkee, India



Module: 7: **Conformal Mapping**
Lecture: 3: **Bilinear transformation**



Bilinear transformation



In this section, we study a special type of transformation called Bilinear Transformation that unifies various well-known transformation.



Bilinear Transformation

- Bilinear transformation is also called Möbius transformation.
- Möbius transformations are named after the geometer August Ferdinand Möbius (1790-1868).
- They are mappings on the extended complex plane (Riemann sphere).
- Every Möbius transformation is a bijective conformal map of the Riemann sphere to itself.



Definition

The transformation given by

$$w = f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0$$

is called a Möbius transformation.



Bilinear Transformation

- In the definition, if a, b, c, d are replaced by $\alpha a, \alpha b, \alpha c, \alpha d$, for some non-zero α , we get the same transformation.
- Hence $w = \frac{az + b}{cz + d}$ does not define a unique map.
- If we choose $\alpha^2 = (ad - bc)^{-1}$, then $(\alpha a)(\alpha d) - (\alpha b)(\alpha c) = 1$.
- Since the properties of the map do not change when we multiply a, b, c, d by a non-zero constant, we can assume $ad - bc = 1$ whenever normalization can make the study simpler.



Bilinear Transformation

- Further $w = \frac{az + b}{cz + d}$ is linear in both z and w and so it is also called bilinear transformation.

- The transformation can also be written as

$$f(z) = \frac{a}{c} - \frac{ad - bc}{c^2(z + \frac{d}{c})}$$

- Hence if we do not impose the condition on a, b, c, d , $f(z)$ reduces to a constant.



Bilinear Transformation

- The constant $ad - bc$ is called the determinant of the transformation.
- The inverse of a bilinear transformation given by

$$z = f^{-1}(w) \frac{-dw + b}{cw - a}$$

is also a bilinear transformation. The determinant of the inverse transformation is $ad - bc$ which is same as $f(z)$.



Bilinear Transformation

- The transformation T associates a unique point of the w plane to each point of the z plane except the point $z = -\frac{d}{c}$ when $c \neq 0$. Also the inverse transformation T^{-1} associates a unique point of the z plane to each point of the w plane except the point $w = \frac{a}{c}$ when $c \neq 0$. These exceptional points $z = -\frac{d}{c}$ and $w = \frac{a}{c}$ are mapped into the point $w = \infty$ and $z = \infty$ respectively.
- Translation, rotation, magnification and inversion are special type of transformations.



- If $ad - bc = 0$, and if $w_1 = \frac{az_1 + b}{cz_1 + d}$ and $w_2 = \frac{az_2 + b}{cz_2 + d}$, then $w_2 - w_1 = 0$. Thus $w_2 = w_1$, hence w is constant. The condition $ad - bc \neq 0$ is essential for bilinear transformation T to set up a one-to-one correspondence between the points of the closed z plane and the closed w plane.



Bilinear Transformation

- From $f(z) = \frac{a}{c} - \frac{ad - bc}{c^2(z + \frac{d}{c})}$, $f(z)$ has a simple pole at $z = -d/c$ while the inverse also has a simple pole at $w = a/c$. Hence

$$f(z) : \mathbb{C} \setminus \{-d/c\} \mapsto \mathbb{C} \setminus \{a/c\}$$

defines a bijective holomorphic function.

- The derivative is given by

$$f'(z) = \frac{ad - bc}{(cz + d)^2}$$

Since $ad - bc \neq 0$, $f'(z) \neq 0$. Hence such transformations are conformal on their domains.



Bilinear Transformation

- If $c = 0$, the transformation $w = f(z) = \frac{az + b}{cz + d}$ is the linear map

$$w = \frac{a}{d}z + \frac{b}{d}$$

So linear map is a special case of a bilinear transformation.

- If $c \neq 0$, this transformation may be written as

$$w = \frac{bc - ad}{c^2(z + \frac{d}{c})} + \frac{a}{c}$$

Here if $Z = z + \frac{d}{c}$, $\zeta = \frac{1}{Z}$, $\tau = \frac{bc - ad}{c^2}\zeta$ and $w = \tau + \frac{a}{c}$ the given bilinear transformation is the resultant of elementary transformations such as translation in Z , inversion in ζ , rotation and magnification in τ and again translation in w .

Bilinear Transformation

- If further $a = d$ and $c = 0$, the map is $w = z + b/d$ which is a pure translation.
- Composition of two bilinear transformation is again a bilinear transformation.
- If C is a circle in the z -plane then the image of C under f is a line if and only if $c \neq 0$ and the pole $z = -d/c$ is on the circle C otherwise it will be a circle in the extended w -plane.
- L is a line in the z -plane then the image of L under f is a line if and only if $c \neq 0$ and the pole $z = -d/c$ is on the line L otherwise it will be a circle in the extended w -plane.



Example

Find the image of the interior of the circle $C : |z - 2| = 2$ under the Möbius transformation

$$w = f(z) = \frac{z}{2z - 8}.$$

Solution. Here f has a pole at $z = 4$ and this point lies on C . Hence the image of the circle is a straight line. To identify this line we have to determine two of its finite points. The image of the points $z = 0$ and $z = 2 + 2i$ are 0 and $-\frac{i}{2}$ as

$$w = f(0) = 0, \quad w = f(2 + 2i) = \frac{2 + 2i}{2(2 + 2i) - 8} = -\frac{i}{2}.$$

Example

Thus the image of C is the imaginary axis in the w plane. Hence the interior of C is, therefore, mapped either onto the right half-plane $\operatorname{Re} w > 0$ or onto the left half-plane $\operatorname{Re} w < 0$. But since the point $z = 2$ lies inside C and

$$w = f(2) = -\frac{1}{2}$$

lies in the left half plane, we conclude that the image of the interior of C is the left half-plane.



Example

Discuss the image of the circle $|z - 2| = 1$ and its interior under the following transformations.

(a) $w = z - 2i$

(b) $w = 3iz$

(c) $w = \frac{z - 4}{z - 3}$



Fixed points of Bilinear Transformation

The fixed point of the bilinear transformation are defined as the points satisfying

$$f(z) = \frac{az + b}{cz + d} = z.$$

- Unless f is the identity transformation $I(z) = z$, that is $a = d \neq 0$ and $b = c = 0$, it is at most quadratic in z .
- Hence $f(z)$ can have at most two (possibly coinciding) fixed points x and y .



Fixed points of Bilinear Transformation

- Suppose $T(z) = \frac{az + b}{cz + d}$, $ad - bc \neq 0$ then a fixed point of $T(z)$ must satisfy the equation

$$\begin{aligned}\frac{az + b}{cz + d} &= z \\ \Rightarrow az + b &= cz^2 + dz \\ \Rightarrow cz^2 + (-a + d)z - b &= 0\end{aligned}$$

- This is a quadratic equation in z and hence a general solution has 2 roots α and β which may be finite or infinite or coincident. If they are unequal distinct points then $(d - a)^2 + 4bc \neq 0$.
- If $(d - a)^2 + 4bc = 0$ there is only one fixed point(both coincide).



Fixed points of Bilinear Transformation

- Suppose $c \neq 0$, $d \neq a$ then

$$\alpha + \beta = \frac{d - a}{c} \text{ is finite, } \alpha\beta = -\frac{b}{c} \text{ is finite.}$$

$\therefore \alpha$ and β are finite.

- Suppose $c = 0$ but $d \neq a$ then the transformation becomes

$$\omega = \frac{az + b}{d} \text{ and if } z \text{ is a fixed point then we have } \frac{az + b}{d} = z \text{ or}$$

$$z = \frac{-b}{a - d}.$$

This is the only one fixed point which is finite.

- Suppose $c = 0$ and $d = a$ the transformation becomes $\omega = z + \frac{b}{a}$.

This being a translation will have only ∞ as the fixed point.

- If for a linear transformation there are more than two fixed points then the equation $cz^2 + (-a + d)z - b = 0$ must be satisfied by more than two values of z .

Fixed points of Bilinear Transformation

Theorem

A linear Transformation with 2 distinct fixed point α and β can be put in the form $\frac{\omega - \alpha}{\omega - \beta} = k \left(\frac{z - \alpha}{z - \beta} \right)$ is constant.

Theorem

A bilinear transformation with 2 coincident fixed points α and α can always be put in the form

$$\frac{1}{w - \alpha} = K + \frac{1}{z - \alpha}.$$



Definition

- A bilinear transformation with a single fixed point α is called parabolic.
- Consider a bilinear transformation with two fixed points α and β .
 - If $|k| = 1$, then the transformation is called Elliptic.
 - If k is real, then the transformation is hyperbolic.
 - If k is neither real, nor $|k| = 1$, then such bilinear transformation is called loxodromic.



Example

Find the fixed points and the normal form of the following bilinear transformations.

1. $w = \frac{z}{2 - z}$

2. $w = \frac{z - 1}{z + 1}$

3. $w = \frac{4z + 3}{2z - 1}$



Cross ratio



In many application we have to construct a linear transformation $w = f(z)$ which maps the given three distinct points z_1, z_2 and z_3 on the boundary of \mathcal{D} to three given distinct points w_1, w_2 and w_3 on the boundary of the image \mathcal{D}' . This is accomplished using the invariance property of the cross-ratio as explained below.



Definition

The cross-ratio of four distinct points z, z_1, z_2 and z_3 is the complex number

$$\frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$



Theorem

If $w = f(z)$ is a bilinear transformation that maps the distinct points z_1, z_2 and z_3 onto the distinct points w_1, w_2 and w_3 respectively then,

$$\frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} = \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1}$$

for all z . In other words, cross-ratio remains invariant under bilinear transformation.



Theorem

The cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points z_1, z_2, z_3, z_4 lie on a circle or on a straight line.

Theorem

Every Möbius transformation maps circles or straight lines into circle or straight lines and inverse points into inverse points.



Example

Find the linear transformation which carries $i, 0, -i$ into $-1, 1, 0$.

Solution

Using the above relation with $z_1 = i, z_2 = 0, z_3 = -i$ and $\omega_1 = -1, \omega_2 = 1, \omega_3 = 0$ we have

$$\begin{aligned}\frac{(z-i)(0+i)}{(z+i)(0-i)} &= \frac{(\omega+1)(1-0)}{(\omega-0)(1+1)} \\ \frac{-z+i}{z+i} - \frac{1}{2} &= \frac{1}{2\omega}\end{aligned}$$

$$\text{Solving } \Rightarrow \omega = \frac{z+i}{-3z+i}.$$



Cross ratio involving the point ∞

- If, for instance, if one of the value is ∞ , say $w_2 = \infty$, then we use the following procedure.
- Replace w_2 by $1/w_2$. This helps in clearing part of the fractions in the numerator and denominator.
- Now substitute $w_2 = 0$. This gives the equation

$$\frac{w - w_1}{w - w_3} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$



Example

Find the Möbius transformation that maps $0, 1, \infty$ to the following respective points.

- $0, i, \infty$
- $0, 1, 2$
- $-i, \infty, 1$
- $-1, \infty, 1$



Clearly the above examples suggest that, to find a bilinear transformation, it is enough to consider three points in the domain and corresponding three points in the range. These points, by means of cross ratio provide the required transformation. This we illustrate in the next example.



Example

To find the Bilinear transformation that maps the upper half plane $\text{Im } z > 0$ onto the open unit disc $|w| < 1$ and the boundary $\text{Im } z = 0$ onto the boundary $|w| = 1$.

Let the Bilinear transformation be

$$T(z) = w = \frac{az + b}{cz + d} \quad ad - bc \neq 0.$$

For this problem, we need to have three points z_1, z_2 and z_3 in the Z -plane and corresponding three points w_1, w_2 and w_3 in the Ω plane. We choose the points $z = 0, z = 1$ and $z = \infty$ in the Z -plane. Then the corresponding points $T(0)$ and $T(1)$ and $T(\infty)$ lie on the boundary $|w| = 1$.

Example

- Note that when $c = 0$, $z = \infty$ gives $T(\infty) = \infty$. But $T(\infty)$ is finite (lies on the unit circle). Hence $c \neq 0$.
- For $z = \infty$ we get $w = a/c$. Since $|w| = 1$, and $c \neq 0$ we get $|a| = |c| \neq 0$.
- Similarly using $z = 0$ we find that $|b| = |d| \neq 0$.
- Hence

$$w = T(z) = \left(\frac{a}{c}\right) \frac{z + (b/a)}{z + (d/c)}.$$

- Since $|a/c| = 1$, we can write this as

$$w = T(z) = e^{i\lambda} \frac{(z - z_0)}{(z - z_1)}, \quad \text{for some } \lambda - \text{real.}$$

Example

- Since $|d/c| = |b/a|$ we get $|z_0| = |z_1|$.
- When $z = 1$ we get $|w| = 1$. Thus

$$|1 - z_1| = |1 - z_0| \implies (1 - z_1)(1 - \bar{z}_1) = (1 - z_0)(1 - \bar{z}_0).$$

- Since $|z_1| = |z_0|$, we get $z_1 \bar{z}_1 = z_0 \bar{z}_0$. Hence this gives

$$z_1 + \bar{z}_1 = z_0 + \bar{z}_0.$$

This means $\operatorname{Re} z_1 = \operatorname{Re} z_0$.

- Further $|z_1| = |z_0|$ gives $z_1 = z_0$ or $z_1 = \bar{z}_0$. But $z_1 = z_0$ gives the transformation as constant. Hence $z_1 = \bar{z}_0$.

Example

Since the points interior to $|w| = 1$ are to be the points above real axis in the Z -plane, the required transformation is

$$w = T(z) = e^{i\lambda} \frac{z - z_0}{z - \bar{z}_0}, \quad \text{for some } \lambda - \text{ real.}$$

Note that the converse is also true.

