NPTEL web course on Complex Analysis

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Module: 7: **Conformal Mapping**
Lecture: 2: **Special transformations**

Conformality at a critical point

Conformality at a critical point

- The basic assumption for conformality is that the derivative is non-zero.
- However all is not lost if the derivative vanishes at some point z_0.
- We now examine the behavior of an analytic function in a neighborhood of a critical point that is at the point where the derivative vanishes.

Conformality at a critical point

- Suppose the derivative of an analytic function $f(z)$ has a zero of order $k-1$ at $z=z_0$.
- We can write

$$f(z)=f(z_0)+\frac{f^k(z_0)}{k!}(z-z_0)^k+\frac{f^{k+1}(z_0)}{(k+1)!}(z-z_0)^{k+1}+\cdots .$$

- This gives

$$\arg[f(z)-f(z_0)]=k\arg(z-z_0)+\arg\left[\frac{f^k(z_0)}{k!}+\frac{f^{k+1}(z_0)}{(k+1)!}(z-z_0)+\cdots\right.$$

Conformality at a critical point

- Suppose θ is the angle that the tangent to a smooth curve at z_0 makes with the positive x-axis and ϕ is the angle the tangent to the image of the curve at $f(z_0)$ makes with the positive u-axis.
- Then as $z \to z_0$, the above equation gives

$$\phi = k\theta + \arg \frac{f^k(z_0)}{k!}.$$

Theorem

Suppose $f(z)$ is analytic at z_0 and that $f'(z)$ has a zero of order $k-1$ at z_0. If two smooth curves intersect at an angle θ in the z-plane, their images intersect at an angle $k\theta$ in the w-plane.

Theorem

An analytic function is conformal at a point if and only if it has a non-zero derivative at the point

Some elementary transformations

Translation: $w = z + \alpha$

- This transformation transforms every point in the z plane as $z + \alpha$ in the direction of the vector α.
- By this transformation the point P is translated along the vector α through a distance $|\alpha|$.
- As it transforms every point of the plane, the image of the region is simply a translation of that region.
- The image have the same shape, size and orientation.

Some elementary transformations

Rotation: $w = e^{i\gamma} z$

- This transformation transforms the image at a point P in the z plane by rotating through the angle γ.
- This rotation is anti-clockwise if $\gamma > 0$ and clockwise if $\gamma < 0$.
- This can be shown analytically as if $z = re^{i\theta}$ and $w = \rho e^{i\phi}$, then $\rho e^{i\phi} = e^{i\gamma}.re^{i\theta} = re^{i(\theta+\gamma)}$

$$\therefore \rho = r \text{ and } \phi = \theta + \gamma$$

- The modulus of w is the same as that of z but the argument of w is increased or decreased by an amount γ according as $\gamma > 0$ or $\gamma < 0$.
- But Geometrically, the two regions are congruent.

Magnification: $w = cz (c > 0)$

By means of this transformation, the figure in the z plane are magnified or contracted according as $c > 1$ or $0 < c < 1$.

Some elementary transformations

Rotation and Magnification: $w = \alpha z$

- If
$$w = \rho e^{i\phi}, \alpha = be^{i\delta}, z = re^{i\theta}$$
- Then the transformation can be written as
$$\rho e^{i\phi} = be^{i\delta}.re^{i\theta} = bre^{i(\theta+\delta)}.$$
- Hence $\rho = br$,$\phi = \theta + \delta$.
- Thus the given transformation transforms the point P in the z plane into the point Q in the w plane whose polar co-ordinates are $(br, \theta + \delta)$.

Note that the transformation $w = bz$ will be only a magnification if b is real, only a rotation if $|b| = 1$ and both magnification and rotation for other cases of b.

Some elementary transformations

Inversion: $w = \dfrac{1}{z}$

- If $w = \rho e^{i\phi}$ and $z = re^{i\theta}$,we get $\rho\, e^{i\phi} = \dfrac{1}{re^{i\theta}} = \dfrac{1}{r} e^{-i\theta} \therefore \rho = \dfrac{1}{r}$ or $|w| = \dfrac{1}{|z|}$ and $\phi = -\theta$.
- Thus points inside the unit circle are mapped into point outside the circle and vice-versa. The points on the circle are reflected in the real axis.

Example

Consider the linear transformation

$$w = (1+i)z + 2 - i$$

and determine the region in the w plane into which the rectangular region bounded by the lines $x = 0, y = 0, x = 1, y = 2$ in the z plane is mapped under this transformation.

Example

Solution: As $z = x + iy$ and $w = u + iv$, we get $u = x - y + 2$ and $v = x + y - 1$.

- The line $x = 0$ is mapped into $u = -y + 2, v = y - 1$ or into $u + v = 1$.
- The line $y = 0$ is mapped into $u = x + 2, v = x - 1$ or into $u - v = 3$.
- The line $x = 1$ is mapped into $u = -y + 3, v = y$ or into $u + v = 3$.
- The line $y = 2$ is mapped into $u = x, v = x + 1$ or into $v - u = 1$.

Hence the given rectangular region in the z plane is mapped into the rectangular region bounded by the lines $u + v = 1, u - v = 3, u + v = 3$ and $v - u = 1$ in the w plane.

Some elementary transformations

Example

From the image in the w plane it is evident that the rectangle in the z plane has gone through magnification by the factor $|1+i| = \sqrt{2}$, rotation by an angle $arg(1+i) = \frac{\pi}{4}$ and finally translated through a distance $|2-i| = \sqrt{5}$ in the direction $2-i$. We have to also note that the origin in the z plane is transformed into the point $2-i$.