NPTEL web course on Complex Analysis

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Module: 7: **Conformal Mapping**
Lecture: 1: **Conformal Mapping**

Conformal Mapping

Introduction

In the previous chapters, the analytic function $f(z)$ was considered in the algebraic sense and its properties were discussed. In this chapter, the analytic function $f(z)$ is considered as a mapping from one domain to another domain and the geometric properties will be discussed. This approach has many applications in applied mathematics.

Conformal Mapping

Definition

Consider an analytic mapping $f : D_1 \to D_2$ given by

$$w = f(z), \qquad z = x + iy \in D_1, \qquad w = u + iv \in D_2.$$

- This is an analytic function mapping the domain D_1 in the xy-plane onto a domain D_2 in the uv-plane.
- We assume that the mapping $w = f(z)$ is a one-to-one correspondence between the points of D_1 and the points of D_2.
- This means for $z_1, z_2 \in D_1$, $f(z_1) = f(z_2)$ only if $z_1 = z_2$.
- By this assumption and fundamental theorem of calculus it is easy to conclude that the derivative $\dfrac{df}{dz}$ is never zero in D_1.

Inverse mapping

Since the given mapping is one-one, inverse exists; that is, with each point of D_2, there can be associated a point of D_1. This point is called pre-image under f. We express this as

$$z = f^{-1}(w).$$

Inverse mapping

- Since only one z is mapped to a particular w, f^{-1} is also a single-valued function.
- f^{-1} is also an analytic function. Its derivative is given by

$$\frac{dz}{dw} = \frac{1}{\frac{dw}{dz}}, \qquad \text{where} \quad w = f(z),$$

- Hence we write

$$\frac{df^{-1}}{dw}(w) = \frac{1}{\frac{df}{dz}(z)}.$$

Example

Let $D_1 = \{z : |\mathrm{Im}\, z| < \pi\}$ and D_2 be the entire plane slit along the negative real axis. Then for $w = e^z$, the inverse is given by $z = Logw$. Further $dw/dz = e^w$ and $dz/dw = 1/z$.

- It is sometimes useful in expressing the given mappings in terms of real variables. Hence $w = f(z)$ can be written as

$$w = u + iv, \qquad u = u(x, y), \quad v = v(x, y)$$

- The inverse mapping $z = f^{-1}(w)$ is written as

$$z = x + iy, \qquad x = x(u, v), \quad y = y(u, v).$$

Conformal Mapping

- In conformal mapping, there are two categories: one is local property and the other is global property.
- Local property means, a property that is satisfied only in a neighborhood of a point z_0 in the domain D.
- Global property is satisfied by all points in the domain D.

Example

Let $w = e^z$. It is one-to-one in any disk of diameter less than 2π and hence it is locally one-to-one. It is not globally one-to-one, since

$$e^{z_1} = e^{z_2} \qquad \text{when} \quad z_1 - z_2 = 2\pi i.$$

Hence this function is locally one-to-one and not globally.

- If there exist a property globally, then it exists locally also. But local existence does not guarantee global existence.

Definition

The concept of extending local properties to global properties is called analytic continuation.

- There are some properties which may exist locally at some point of a domain and may not exist at some other points.

Example

Let $w = f(z) = z^2$. Consider any open set that contains origin. Assume that in that open set, $z_1 = -z_2$. Then $z_1^2 = z_2^2$. Thus f is not one-to-one. If we exclude the origin, we can a neighborhood of each point where $f(z) = z^2$ is one-to-one. Thus, $f(z) = z^2$ is one-to-one at every point other than the origin.

- The reason for such behavior can be explained using the following result.

Theorem

If f is analytic at z_0 and $f'(z_0) \neq 0$, then there is an open disk D centered at z_0 such that f is one-to-one in D.

- This theorem guarantees that an analytic function is locally one-to-one at points where its derivative does not vanish.

Conformal Mapping

- One of the interesting local property is conformality.
- We need some preliminaries before defining this concept.

Conformal Mapping

- A straight line passing through the point z_0 in the complex plane can be expressed as

$$l(r) = z_0 + re^{i\theta}, \quad r \in \mathbb{R}$$

  where θ is the angle (measured in radians) between the positive real axis and the line $l(r)$.
- Suppose $z(t)$ is a smooth curve, that is it has a tangent at each point.
- Let f be an analytic function defined on $z(t)$.
- Our aim is to compare the inclination of the tangent to $z(t)$ at a point $z_0 = z(t_0)$ with the inclination of the tangent to the image curve at the image point $f(z_0)$.

Conformal Mapping

- The image of $z(t)$ under f

$$w = w(t) = f(z(t))$$

  is again a smooth curve.
- The tangent T to the image curve $w(t)$ at the point $w_0 = f(z_0)$ is

$$T = w'(t_0) = f'(z_0)z'(t_0).$$

Conformal Mapping

- If the angle of inclination of the tangent to the curve $z(t)$ is $\phi = \arg z'(t_0)$, then the angle of inclination of T at $f(z_0)$ is $\arg T = \arg f'(z_0) + \phi$.
- Hence the angle of inclination of the tangent to $z(t)$ at z_0 is rotated through the angle $\arg f'(z_0)$ to obtain the angle of inclination of the tangent T to $w(t)$ at the image point w_0.
- Hence the difference between the tangent to a curve at a point and the tangent of the image curve at the image point depends only on the derivative of the function at the point.
- Of course we have to assume here $f'(z_0) \neq 0$ so that $\arg f'(z_0)$ has a meaning.

Conformal Mapping

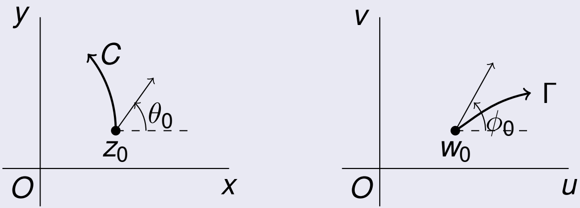
- If two curves intersect at a point, then the angle between these two curves is defined as the angle between the tangents to these curves.
- Suppose now that $z_1(t)$ and $z_2(t)$ are two smooth curves intersecting at the point z_0 and their images under an analytic function f be $w_1(t)$ and $w_2(t)$.
- Further suppose the tangents to $z_1(t)$ and $z_2(t)$ at z_0 make angles θ_1 and θ_2 respectively with the positive real axis.
- Then the angle between $z_1(t)$ and $z_2(t)$ at z_0, , measured from $z_1(t)$ to $z_2(t)$ is $\theta_2 - \theta_1$.

- Using $\arg T = \arg f'(z_0) + \phi$, the angle between $w_1(t)$ and $w_2(t)$ measured from $w_1(t)$ to $w_2(t)$ is

$$[\theta_2 + \arg f'(z_0)] - [\theta_1 + \arg f'(z_0)] = \theta_2 - \theta_1.$$

- Hence $f(z)$ preserves the angle between the two curve at z_0. It is worth repeating that the underlying assumption here is that $f'(z_0) \neq 0$.

Conformal Mapping



Theorem

Suppose $f(z)$ is analytic at z_0 with $f'(z) \neq 0$. Let C_1 and C_2 be smooth curves in the z-plane that intersect at z_0 with C_1' and C_2' being the images of C_1 and C_2 respectively. Then the angle between C_1 and C_2, measured from C_1 to C_2 is equal to the angle between C_1' and C_2' measured from C_1' to C_2'.

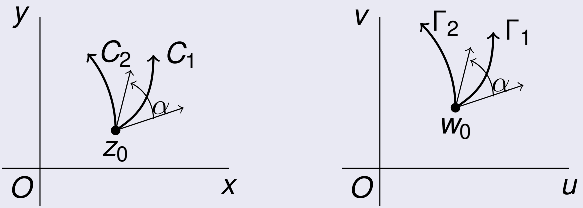
Conformal Mapping

Definition

A function that preserves both the magnitude and the orientation of the angle is said to be **conformal**.

- The above Theorem says that an analytic function is conformal at all points where the derivative is non-zero.
- In the above discussion, the magnitude is $\theta_2 - \theta_1$ and the orientation is in the sense that the angle in the z-plane is measured from C_1 to C_2, and from C_1' to C_2' (their respective images) in the w-plane.

Conformal Mapping



Definition

A function that preserves the magnitude but not the orientation is said to be **isogonal**.

Example

The simplest example is $f(z) = \bar{z}$.

- It maps the positive real axis and the positive imaginary axis to the positive real axis and negative imaginary axis respectively.
- The curves intersect at right angles in both the planes, but a counterclockwise orientation gives way to a clockwise orientation in the image.

Conformal Mapping

- It may be noted $f(z) = \bar{z}$ is not analytic.
- Non-analytic functions do not necessarily need to be isogonal.
- Similarly all analytic functions also do not need to be conformal.
- In fact we characterize those analytic functions which are conformal on a domain by the following theorem

Theorem

If $f(z)$ is analytic and 1-1 in a domain $\mathcal{D}$, then $f'(z) \neq 0$ in $\mathcal{D}$.

- It may be noted that non-vanishing of a derivative on a domain is necessary but not a sufficient condition for the function to be one-one.
- The sufficient condition for an analytic function to be 1-1 in a simply connected domain is that it should be 1-1 on its boundary.