NPTEL web course on Complex Analysis

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Module: 2: **Functions of a Complex Variable**
Lecture: 5: **Cauchy-Riemann equations**

Cauchy Riemann equations

Cauchy-Riemann equations

Consequence of analyticity

Let $D \in \mathbb{C}$ and f be analytic in D. Then for $(x_0, y_0) \in D$,

$$f^{'}(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exist and this limit is unique.

Cauchy-Riemann equations

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exist and this limit is unique.

Note.

This implies approaching $\Delta z \to 0$ through any direction gives the same limit.

Cauchy-Riemann equations

Consequence of analyticity

Let $f(z) = u + iv = u(x, y) + iv(x, y)$. Then

$$f'(z_0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x_0 + iy_0 + \Delta x_0 + i\Delta y_0) - f(x_0 + iy_0)}{\Delta x_0 + i\Delta y_0}. \qquad (1)$$

We consider two cases, wherein the limits are approached through the coordinate axes.

Cauchy-Riemann equations

Consequence of analyticity

(i) Allow $\Delta z \to 0$ through x axis, i.e. $\Delta x \to 0$. This implies $f'(z_0)$

$$= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

$$= u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Consequence of analyticity

(ii) Allow $\Delta z \to 0$ through y axis, i.e. $\Delta y \to 0$. This implies $f^{'}(z_0)$

$$= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}$$

$$+ i \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y}$$

$$= -iu_y(x_0, y_0) + v_y(x_0, y_0).$$

Cauchy-Riemann equations

Consequence of analyticity

Since the limit of both these cases are equal,

$$f^{'}(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = -iu_y(x_0, y_0) + v_y(x_0, y_0).$$

Cauchy-Riemann equations

Consequence of analyticity

Since the limit of both these cases are equal,

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = -iu_y(x_0, y_0) + v_y(x_0, y_0).$$

Comparing the real and imaginary parts, we have

$$\begin{aligned} u_x = v_y \quad &\text{and} \quad v_x = -u_y \quad \text{or} \\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad &\text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \end{aligned} \tag{2}$$

(2) is called Cauchy-Riemann equation, or C-R equations.

Necessary condition

Theorem

Let $f = u + iv$ be analytic and $f'(z)$ exist at a point z_0 in a domain D. Then the first order partial derivatives u_x, u_y exist at $z_0 = (x_0, y_0)$ and they must satisfy C-R equations given by (2) at $z = z_0$.

Cauchy-Riemann equations

Necessary and sufficient condition

Remark

If f is analytic in a domain D. Then $f = u + iv$ satisfies C-R equations. The converse is not true; i.e., if $f = u + iv$ satisfies C-R equations then f need not be analytic.

Sufficient condition

Example

$$f(x,y) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

satisfies C-R equations at origin, but not analytic.

Sufficient condition

Example

Explanation.

- Writing $f(x, y) = u(x, y) + iv(x, y)$ we get

$$u := u(x, y) = \frac{x^3 - y^3}{x^2 + y^2}, \qquad v := v(x, y) = \frac{x^3 + y^3}{x^2 + y^2}.$$

- For $z \neq 0$, since u and v are rational functions with non-zero denominators.
- Hence u and v are continuous when $z \neq 0$.

Sufficient condition

Example

- Further, taking $x = r\cos\theta$, $y = r\sin\theta$ gives

$$u = r(\cos^3\theta - \sin^3\theta), \qquad v = r(\cos^3\theta + \sin^3\theta),$$

  and the limit $z \to 0$ implies $r \to 0$.
- Thus the limiting value of u and v is equal to 0. Hence $f(x, y)$ is continuous for all values of (x, y).

Cauchy-Riemann equations

Sufficient condition

Example

Now

$$u_x = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{x - 0}{x} = 1$$
$$u_y = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{-y - 0}{y} = -1$$
$$v_x = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{x - 0}{x} = 1$$
$$v_y = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{y - 0}{y} = 1.$$

Sufficient condition

Example

Now

$$u_x = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$$
$$u_y = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{-y-0}{y} = -1$$
$$v_x = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$$
$$v_y = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{y-0}{y} = 1.$$

Hence $u_x = v_y$ and $v_x = -u_y$ and the C-R equations are satisfied.

Sufficient condition

Example

- But, we have

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{(\Delta x)^3 - (\Delta y)^3 + i\left((\Delta x)^3 + (\Delta y)^3\right)}{(\Delta x)^2 + (\Delta y)^2} \frac{1}{(\Delta x + i\Delta y)}.$$

- If $\Delta z \to 0$ along x axis, i.e., $\Delta y \to 0$ then $f'(0) = 1 + i$.
- Whereas if $\Delta z \to 0$ along the curve $y = x$, then $f'(0) = (1+i)/2$. Since the limits are different, $f'(0)$ does not exist.

Sufficient condition

When existence of C-R equations implies analyticity of f?

Cauchy-Riemann equations

Sufficient condition

Theorem

If f is defined at a point $z_0 = (x_0, y_0)$ and in the neighbourhood of z_0. Let the following conditions hold:

1. *f is continuous at a point (x_0, y_0) and in the neighbourhood of z_0.*
2. *first order partial derivative exist everywhere in the neighbourhood of z_0.*
3. *first order partial derivative are continuous at (x_0, y_0).*
4. *first order partial derivative satisfies C-R equations (2) at (x_0, y_0).*

Then f is analytic at (x_0, y_0).

Cauchy-Riemann equations

Sufficient condition

- In the previous example, first order partial derivative are not continuous at $(0,0)$. Hence f is not continuous.
- Taking $x = r\cos\theta$ and $y = r\sin\theta$ gives
  $u(x,y) = u(r\cos\theta, r\sin\theta) = u(r,\theta)$ and
  $v(x,y) = v(r\cos\theta, r\sin\theta) = v(r,\theta)$.

Cauchy-Riemann equations

Polar form

Theorem

If $f(z) = u(r, \theta) + iv(r, \theta)$ *then C-R equations are*

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad v_r = -\frac{1}{r} u_\theta. \tag{3}$$

Cauchy-Riemann equations

Polar form

Theorem

If $f(z) = u(r,\theta) + iv(r,\theta)$ *then C-R equations are*

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad v_r = -\frac{1}{r} u_\theta. \tag{3}$$

Hint: This can be proved by writing $x = r\cos\theta$, $y = r\sin\theta$ *and using the C-R equations given by (2).*