NPTEL web course on Complex Analysis

A. Swaminathan
I.I.T. Roorkee, India

and

V.K. Katiyar
I.I.T. Roorkee, India

Module: 4: **Complex Integration**
Lecture: 3: **Cauchy Integral Formula**

Theorem on antiderivative

Theorem

Let f be continuous in D and has antiderivative F throughout D , i.e. $\frac{d}{dz}F = f$ in D. Then for any closed contour C in D

$$\int_c f(z)dz = 0.$$

Theorem on antiderivative

Proof

- From the previous result,

$$\int_C f(z)dz = F(z_T) - F(z_I).$$

- Since C is closed, $z_T = z_I$.
- This means $\int_C f(z)dz = 0$.

Proof

- From the previous result,

$$\int_c f(z)dz = F(z_T) - F(z_I).$$

- Since C is closed, $z_T = z_I$.
- This means $\int_c f(z)dz = 0$.

Remark

This is alternative to Cauchy fundamental theorem.

Equivalent statements

Theorem

Let $f(z)$ be continuous in a domain D. Then the following are equivalent.

(i) *f has antiderivative.*

(ii) *For every closed curve c, $\int_c f(z)dz = 0$.*

(iii) *For two curves Γ_1 and Γ_2, joining the points z_1 and z_2*

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$$

Equivalent statements

Proof

- (i) $\Rightarrow$ (ii) is the previous result.
- For (ii) $\Rightarrow$ (iii), let Γ_1 & Γ_2 be taken with positive orientation.
- Define $c = \Gamma_1 \cup \Gamma_2'$ where $\Gamma_2' = -\Gamma_2$.
- c is positive oriented.
- Hence by (ii) $\int_c f(z)dz = 0$.

Equivalent statements

Proof

- Thus $\int_c f(z)dz = 0$ implies

$$\begin{aligned} 0 &= \int_{\Gamma_1 \cup \Gamma_2'} f(z)dz \\ &= \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2'} f(z)dz \\ &= \int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz. \end{aligned}$$

- Therefore $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$, which is (iii)

Equivalent statements

Proof

For (iii) $\Rightarrow$ (i)

- Let (iii) be true.
- To prove that $\exists$ F such that F is analytic and $\dfrac{d}{dz}F = f$ for all z in D.
- Define $F(z) = \int_{z_0}^{z} f(s)ds$, for some fixed z_0.
- Then $F(z)$ is well defined.
- Now

$$F(z + \Delta z) = \int_{z_0}^{z+\Delta z} f(s)ds.$$

Proof

- Let $\Delta z = \int_z^{z+\Delta z} ds$,
- $\implies f(z)\Delta z = \int_z^{z+\Delta z} f(z)ds$.
- This means,

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)ds - \int_z^{z+\Delta z} f(s)ds$$
$$= \int_z^{z+\Delta z} f(s)ds.$$

Equivalent statements

Proof

Hence,

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right| = \frac{1}{|\Delta z|}\left|F(z+\Delta z)-F(z)-f(z)\Delta z\right|$$
$$= \frac{1}{|\Delta z|}\left|\int_z^{z+\Delta z} f(s)ds - \int_z^{z+\Delta z} f(z)ds\right|$$
$$= \frac{1}{|\Delta z|}\left|\int_z^{z+\Delta z}[f(s)-f(z)]ds\right|$$
$$\leq \frac{1}{|\Delta z|}\int_z^{z+\Delta z}|f(s)-f(z)|\,|ds|$$

Equivalent statements

Proof

- Since f is continuous , for given $\epsilon > 0, \exists\, \delta > 0$ such that $|f(s) - f(z)| < \epsilon$ whenever $|s - z| < \delta$.
- Hence, right hand side of the previous expression has $< \epsilon$.
- This means, for all s close to z , $\dfrac{d}{dz}F(z) \equiv f(z)$.
- Since this is true for all z , in that neighbourhood, $F(z)$ is analytic at z.

Example

To evaluate $\int_\Gamma \frac{dz}{z - z_0}$, where $\Gamma = \{z : |z - z_0| = r\}$ traverses twice.
Here $z - z_0 = re^{i\theta}$, $0 \leq \theta \leq 4\pi : \theta = 2\phi \Rightarrow 0 \leq \phi \leq 2\pi$, $z - z_0 = re^{i\theta}$
Hence

$$I = \int_0^{2\pi} \frac{2ire^{ir2\phi}}{re^{i2\phi}} d\phi = 4\pi i.$$

Cauchy Integral Formula

Theorem

Let f be analytic in a region R enclosed by a simple closed contour C. If $z_0 \in$ int C, (interior of C), then for any $z \in D$

$$\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0).$$

Cauchy Integral formula

Proof

$$\int_C \frac{dz}{z - z_0} = 2\pi i \Longrightarrow \int_C \frac{f(z_0)}{z - z_0} dz = 2\pi i f(z_0)$$

Consider

$$I = \frac{1}{2\pi i} \int_c \frac{f(z) - f(z_0)}{z - z_0} dz.$$

- Since f is analytic in D , for any z in the neighbourhood of z_0 , $|f(z) - f(z_0)| < \epsilon$ whenever z in a disk of radius ρ centered at z_0.

Cauchy Integral formula

Proof

Thus

$$\begin{aligned} |I| &= \left| \frac{1}{2\pi i} \int_C \frac{f(z) - f(z_0)}{z - z_0} dz \right| \\ &\leq \frac{1}{2\pi} \int_C \frac{|f(z) - f(z_0)|}{|z - z_0|} |dz|. \\ &< \frac{\epsilon}{2\pi\rho} (2\pi\rho) = \epsilon. \end{aligned}$$

- The result is true by replacing the disk $z : |z - z_0| < \rho$ by any contour c that lies entirely inside the disc (ρ, z_0) which is the region R.

Example

Question: Find $\int_{\Gamma} \frac{g(z)}{z(z-4)} dz$, where $\Gamma = \{z : |z| < 2\}$.

Answer.

- Let $f(z) = \frac{g(z)}{z}$. Then $f \notin \mathcal{A}$ in $|z| < 2$.
- Hence Cauchy Integral Formula cannot be applied.
- Therefore, suppose that $f(z) = \frac{g(z)}{z-4}$.
- Then $f \in \mathcal{A}$ in $|z| < 2$
- Hence by Cauchy Integral Formula,

$$\int_{\Gamma} \frac{g(z)}{z(z-4)} dz = \int_{\Gamma} \frac{f(z)}{z-0} dz = 2\pi i f(0) = -\frac{\pi i}{2} g(0).$$

Consequence of Cauchy Integral formula

Poisson Integral formula

Theorem

Let $f \in A$ in $|z| < \rho$ and $z = re^{i\theta}$ in a domain D that contains $|z| < \rho$. Then

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})}{R^2 - 2\pi R \cos(\theta - \phi) + r^2} \, d\phi,$$

where $0 < R < \rho$.

Further details regarding this result will be discussed in the last chapter.