NPTEL web course on Complex Analysis

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Module: 2: **Functions of a Complex Variable**
Lecture: 4: **Analytic functions**

Derivatives

A small note

- So far, we have seen the mapping of functions that takes values from Z-plane (xy-plane) to the Ω-plane(uv-plane). In the Ω-plane, u and v are functions of x and y.

$$w = u + iv \in \Omega, \qquad u = u(x, y), \quad v = v(x, y),$$
$$x + iy \in Z \quad (xy - \text{plane}).$$

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- In the one-variable real case, the concept that follows continuity is derivative. Here, before the concept of differentiability, we consider the concept of partial derivatives which is similar to the two-variable real case.

Partial Derivatives

Definition

Let $u(x, y)$ be defined at all points in the neighbourhood of (x_0, y_0). Then we define the partial derivative of u with respect to x and y, respectively, as

$$u_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$
$$u_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}$$

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Notation

Some authors prefer to write the notation $u_x = \dfrac{\partial u}{\partial x}$.

Partial Derivatives

Second order partial derivatives

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}, \qquad u_{yy} = \frac{\partial^2 u}{\partial y^2},$$

$$u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \qquad u_{yx} = \frac{\partial^2 u}{\partial y \partial x},$$

where u_{xy} and u_{yx} are called second order mixed partial derivatives.

Note

Assume that, for the function $u(x, y)$, the first order partial derivatives u_x, u_y and the second order partial derivatives u_{xx}, u_{yy}, u_{xy} and u_{yx} exist at (x_0, y_0). Further if all these partial derivatives are continuous in the neighbourhood (x_0, y_0), then the mixed partial derivatives are equal at (x_0, y_0); i.e.,

$$u_{xy}(x_0, y_0) = u_{yx}(x_0, y_0).$$

Example

Let

$$u(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{for} \quad (x,y) \neq (0,0) \\ 0 & \text{for} \quad (x,y) = (0,0). \end{cases}$$

For this function, $u_{xy}(0,0) = -1$ and $u_{yx}(0,0) = 1$, because the mixed partial derivatives are not continuous at $(0,0)$.

Derivatives

Definition

Let $f : D_1 \to D_2$. Let $z_0 \in D$. f is differentiable at z_0 if

$$\lim_{\Delta \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists. This derivative is denoted as $f^{'}(z_0)$.

Note.

If $f^{'}(z_0)$ exists for each $z_0 \in D$, then f is said to be differentiable in D.

Derivatives

Example

Question. Find the derivative of $f(z) = z^n$.
Answer. Using binomial theorem, we write

$$\frac{(z+\Delta z)^n - z^n}{\delta z} = \frac{nz^{n-1}\Delta z + \frac{n(n-1)}{2} z^{n-2}(\Delta z)^2 + \cdots + (\Delta z)^n}{\Delta z}$$

$$= nz^{n-1} + \frac{n(n-1)}{2} z^{n-2}(\Delta z) + \cdots + (\Delta z)^{n-1}.$$

Thus

$$f'(z) = \lim_{\Delta z \to 0} \left(nz^{n-1} + \frac{n(n-1)}{2} z^{n-2}(\Delta z) + \cdots + (\Delta z)^{n-1} \right) = nz^{n-1}.$$

Note.

The above example is similar to the real-variable case. Hence the following results can be adapted from the real variable calculus.

Derivatives

Results

Theorem

If f and g are differentiable then

$$\begin{aligned}
(f \pm g)'(z) &= f'(z) \pm g'(z),\\
(cf)'(z) &= cf'(z) \qquad \text{for some non-zero real constant} \quad c,\\
(fg)'(z) &= f(z)g'(z) + f'(z)g(z),\\
\left(\frac{f}{g}\right)'(z) &= \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}, \qquad \text{if} \qquad g(z) \neq 0.
\end{aligned}$$

Chain Rule

Theorem

Let $f(z)$ be a differentiable function of z and $w = f(z)$. If $g(w)$ is a differentiable function of w, then the composition function $g(f(z))$ is also differentiable and

$$g'(w) = (g(f(z)))' = g'(f(z))f'(z).$$

Analytic functions

A topological behaviour of a complex valued function, obtained from the concept of its derivative, is called Analytic function.

Analytic function

Definition

Let $f : D_1 \to D_2$. Let f be differentiable at z_0 and also in the neighborhood of z_0. Then f is said to be analytic at z_0. If this is true for each $z_0 \in D_1$ then f is analytic in D_1.

Analytic function

Example

(i). $f(z) = z$ is analytic at everywhere.

(ii). All polynomials are analytic in $\mathbb{C}$.

(iii). $f(z) = \dfrac{z}{1-z}$ is not analytic at $z = 1$.

Non-Analytic function

Example

Question. To check if $f(z) = \bar{z}$ is analytic at origin.
Answer. First we find the derivative of $f(z) = \bar{z}$ at origin. For this, we need to find

$$\lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} \qquad \text{exists.}$$

This implies

$$\lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z} - 0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{(\Delta x, \Delta y) \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

Now we consider two different cases.

non-Analytic function

Example

case(i): Let $\Delta x \to 0$ first. Then,

$$\lim_{\Delta y \to 0} \left(\lim_{\Delta x \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \right) = \lim_{\Delta y \to 0} (-1) = -1.$$

case(ii): Let $\Delta y \to 0$ first. Then,

$$\lim_{\Delta x \to 0} \left(\lim_{\Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \right) = \lim_{\Delta x \to 0} (1) = 1.$$

Since taking the limit in different ways leads to different values , $f(z) = \overline{z}$ is not differentiable at origin and hence not analytic.

non-Analytic function

Example

In general $f(z) = \overline{z}$ is nowhere differentiable and hence is not analytic anywhere in the complex plane $\mathbb{C}$.

non-Analytic function

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Remark

Since $f(z) = \overline{z}$ is not analytic, any $f(z)$ that can be written in terms of $\overline{z}$ is not analytic. Since

$$2\mathrm{Re}\, z = z + \overline{z}, \qquad 2i\mathrm{Im}\, z = z - \overline{z},$$

$f(z)$ is not analytic if it can be written in the form consisting of $\overline{z}$, Re z, Im z.

A consequence

One of the important consequences of analytic functions is the following result.

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Theorem

Let f be analytic in a open connected set (domain) D and $f'(z) = 0$ everywhere in D. Then f is constant throughout D.

Remark

Note that in the previous result, the connectedness of the domain is essential. For example, define $f(z)$ by

$$f(z) = \begin{cases} \alpha & \text{if} \quad |z| < r_1 \\ \beta & \text{if} \quad |z| > r_1 + 2, \end{cases}$$

for some constant α and $r_1 > 0$. Then also $f'(z) = 0$. But $f(z)$ is not constant. Here the domain where $f(z)$ is defined is not connected.

Analytic function

Theorem

Let f be analytic in a domain D_1 and g be analytic in a domain D_2, then their sum, difference and product are analytic in the domain $D_1 \cap D_2$. Similarly their quotient f/g is analytic in $D_1 \cap D_2$, if $g(z) \neq 0$ for $z \in D_1 \cap D_2$.

Choice of domain for resultant function

Remark

The domain for the resultant function of two analytic function should be chosen very carefully. For the composition of two analytic functions which follows the chain rule, an example is given below to illustrate the choice of the domain.

Choice of domain for resultant function

Example

- Define $f(z) = z^2$.
- Clearly this is an entire function (analytic in the whole complex plane).
- Define $g(z) = z^{1/2}$. This is a multi-valued function.
- Converting this function into polar form and restricting the domain of definition as

$$g(z) = z^{1/2} = \sqrt{\rho} e^{i\phi/2}, \qquad (\rho > 0, \quad -\pi < \phi < \pi)$$

it can be seen that this function has a derivative at each point of z in the domain of definition and $g'(z) = 1/2g(z)$.

Choice of domain for resultant function

Example

- Hence this function is analytic everywhere in the given domain of definition.
- For the composition $g(f(z))$, writing $w = f(z) = re^{i\theta}$, the function is defined as

$$g(f(z)) = g(w) = w^{1/2} = \sqrt{r}e^{i\theta/2}, \qquad (r > 0, \quad -\pi < \theta < \pi).$$

- Even though, the domain of f is $\mathbb{C}$, we should restrict its domain such that the range of f is contained in the domain on which g is defined.

Analytic function

Choice of domain for resultant function

Example

- Hence the largest possible domain in which f can be defined so that the range of f should lie in the domain $r > 0, -\pi < \theta < \pi$.
- For this purpose, we write $w = \rho^2 e^{i2\phi}$. This gives $-\pi < \phi < \pi$ when $-\pi/2 < \phi < \pi/2$.
- Thus $r = \rho^2$ and $\theta = 2\phi$, where $r > 0$ and $-\pi < \theta < \pi$.
- Hence the plane $\rho > 0, -\pi/2 < \phi < \pi/2$ is the largest possible domain that can be taken for the definition of f.

Choice of domain for resultant function

Example

- This domain gives the analyticity of $f(g(z))$ from the analyticity of f and g in the respective domain.
- Hence

$$g(f(z)) = g(w) = \sqrt{\rho^2} e^{i2\phi/2} = \rho e^{i\phi} = z,$$

for any point z in the domain $\rho > 0$, $-\pi/2 < \phi < \pi/2$.